

SOME TOPICS IN THERMODYNAMICS AND QUANTUM MECHANICS

ROBERT CARROLL
UNIVERSITY OF ILLINOIS, URBANA, IL 61801

CONTENTS

1. BACKGROUND	1
2. FRACTIONAL SCHRÖDINGER EQUATION	7
3. Q-DEFORMED AND FRACTIONAL CALCULUS	12
4. THE Q EXPONENTIAL FAMILY	15
5. TD AND FRACTALS	17
6. Q-FISHER AND TSALLIS	23
References	26

1. BACKGROUND

The first theme here is to use fractional calculus (modified Riemann-Liouville - MRL (d'apres Jumarie [20, 43]) and q-deformed calculus (d'apres [12, 37]) to develop the Schrödinger equation (SE) in various guises. In particular we obtain fractional and q-deformed quantum potentials (QP) and this context will have interaction with with e.g. q-Fisher information and various ideas of entropy (e.g Renyi, Tsallis, etc.) for which we provide a very brief survey (cf. e.g. [26, 27, 29, 32, 33, 42, 75, 80]). Hopefully this should help in developing thermodynamic (TD) connections with fractal situations and quantum mechanics (QM). We began this in [14] where a fractional QP was outlined for a very elementary fractional derivative of V . Kobolev (cf. [48]) and in fact that treatment can be slightly refined, leading to the same conclusion (cf. Section 2). We do not review here the various fractional calculi (cf. [20, 37, 43, 45, 46] but simply begin with MRL as in [20, 43]. Later we will expand this when discussing the q-deformed calculus (as in [37]).

We go to the directly to the modified RL (MRL) integral of Jumarie following [20, 43]. Thus from [20], pp. 1-69, with some stated and some

Date: November, 2012.
email: rcarroll@math.uiuc.edu.

unstated hypotheses, define **(1A)** $F_w(h)f(x) = f(x+h)$ with fractional difference

$$(1.1) \quad \Delta^\alpha f(x) = (F_w - 1)^\alpha f(x) = \sum_0^\infty (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h]$$

and fractional derivative

$$(1.2) \quad f^{(\alpha)}(x) = \lim_{h \rightarrow 0^+} \frac{\Delta^\alpha [f(x) - f(0)]}{h^\alpha}$$

While the RL involves

$$(1.3) \quad f^{(\alpha)} = \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha-1} f(\xi) d\xi, \quad (\alpha < 0)$$

Jumarie uses the MRL ($0 < \alpha < 1$)

$$(1.4) \quad f^{(\alpha)}(x) = [f^{\alpha-1}(x)]' = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha} [f(\xi) - f(0)] d\xi$$

$$(1.5) \quad f^{(\alpha)}(x) = [f^{(\alpha-n)}(x)]^{(n)} = \frac{1}{\Gamma(1-(\alpha-n))} \frac{d^{n+1}}{dx^{n+1}} \int_0^x (x-\xi)^{-\alpha-n} (f(\xi) - f(0)) d\xi$$

for $n \leq \alpha < n+1$. Laplace transforms are useful in proving some of the results in fractional calculus and we refer to [20, 43] for details (some proofs are given here but generally we are simply stating results in order to have a quick overview of the theory. If $f(x)$ is differentiable this is equivalent to the Liouville-Dzhrbashian-Caputo definition

$$(1.6) \quad f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x - \xi)^{-\alpha} f'(\xi) d\xi \quad (0 < \alpha < 1)$$

In general, to avoid non-commutativity problems one defines **(1B)** $D^{a+b} = D^{\max(a,b)} D^{\min(a,b)}$. A generalized Taylor expansion of fractional order is given by

$$(1.7) \quad f(x+h) = \sum_0^\infty \frac{f^{\alpha k}(x)}{(\alpha k)!} h^{\alpha k} \quad (0 < \alpha \leq 1)$$

where $f^{(\alpha k)}(x) = D^\alpha D^\alpha \dots D^\alpha f(x)$ and **(1C)** $\Gamma(1 + \alpha k) = (\alpha k)!$. We note also that

$$(1.8) \quad F_w(h) = E_\alpha(h^\alpha D_x^\alpha); \quad E_\alpha(u) = \sum_0^\infty \frac{u^k}{(\alpha k)!}$$

The equation (1.7) provides the useful relation **(1D)** $d^\alpha f = \Gamma(1+\alpha)df$, $0 < \alpha < 1$ or equivalently **(1E)** $\Delta^\alpha f \simeq \alpha! \Delta f$ which holds for non-differentiable functions only. One can also write from this

$$(1.9) \quad D^{n+\theta} x^\gamma = \Gamma(\gamma+1) \Gamma^{-1}(\gamma+1-n-\theta) x^{\gamma-n-\theta} \quad (0 < \theta < 1);$$

$$(u(x)v(x))^{(\alpha)} = u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x); [f(u(x))]^{(\alpha)} = f_u^{(\alpha)}(u)(u'_x)^\alpha$$

In this spirit, if both u and v are non-differentiable, one has $\alpha!d(uv) = u d^\alpha v + v d^\alpha u$ and (1.9C) is a consequence of

$$(1.10) \quad \frac{d^\alpha f(u)}{dx^\alpha} = \frac{d^\alpha f}{du^\alpha} \left(\frac{du}{dx} \right)^\alpha$$

Note here that

$$(1.11) \quad d^\alpha f = \alpha! df \iff f(u) \text{ is non-differentiable}$$

Further we write

$$(1.12) \quad \frac{d^\alpha u}{du^\alpha} = \frac{1}{(1-\alpha)!} u^{1-\alpha} \Rightarrow$$

$$(du)^\alpha = (1-\alpha)u^{\alpha-1}d^\alpha u \Rightarrow (u'_x(x))^\alpha = (1-\alpha)!u^{\alpha-1}u_x^{(\alpha)}(x)$$

Note that $u_x^{(\alpha)}(x)$ may exist while $u'(x)$ is undefined but if $u'(x)$ exists then $u^{(\alpha)}(x)$ exists with relations as in (1.12). If $f(u)$ is differentiable and $u(x)$ is differentiable then

$$(1.13) \quad (f[u(x)])^{(\alpha)} = \frac{1}{(1-\alpha)!} [f(u)]^{1-\alpha} \left(\frac{df}{du} \right)^\alpha \left(\frac{du}{dx} \right)^\alpha$$

If both $f(u)$ and $u(x)$ have fractional derivatives then

$$(1.14) \quad (f[u(x)])^{(\alpha)} = (1-\alpha)!u^{\alpha-1}(f(u))^{1-\alpha}f_u^{(\alpha)}(u)u_x^{(\alpha)}(x)$$

To see this write

$$(1.15) \quad \frac{d^\alpha f(u(x))}{dx^\alpha} = \frac{d^u f}{d^\alpha u} \frac{d^\alpha u}{dx^\alpha} = \frac{d^\alpha f}{du^\alpha} \frac{du^\alpha}{d^\alpha u} \frac{d^\alpha u}{dx^\alpha}$$

and use (1.12). The integration with respect to $(dx)^\alpha$ is defined as the solution of the fractional equation **(1F)** $dy = f(x)(dx)^\alpha$, $x \geq 0$, $y(0) = 0$, $0 < \alpha < 1$ which is provided via the equation

$$(1.16) \quad y = \int_0^x f(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi \quad (0 < \alpha \leq 1)$$

More generally

$$(1.17) \quad \int_a^x f(\xi)(d\xi)^\alpha = \alpha \int_a^x (x-\xi)^{\alpha-1} f(\xi-a) d\xi, \quad (0 < \alpha \leq 1)$$

and there is an integration by parts formula

$$(1.18) \quad \int_a^b u^{(\alpha)}(x)v(x)(dx)^\alpha = \alpha! [u(x)v(x)]_a^b - \int_a^b u(x)v^{(\alpha)}(x)(dx)^\alpha$$

If $y = g(x)$ is a non-decreasing differentiable function then

$$(1.19) \quad \int f(y)(dy)^\alpha = \int f(g(x))(g'(x))^\alpha (dx)^\alpha \quad 0 < \alpha < 1$$

and when $g(x)$ has a positive fractional derivative of order β , $0 < \alpha, \beta < 1$ there results

$$(1.20) \quad \int f(y)(dt)^\alpha = \Gamma^{-\alpha}(1 + \beta) \int f(g(x)(g^{(\beta)}(x)))^\alpha (dx)^{\alpha\beta}$$

A few special formulas are obtained by setting $f(x) = x^\gamma$ in (4.29) to get

$$(1.21) \quad \int_0^x \xi^\gamma (d\xi)^\alpha = \frac{\Gamma(\alpha + 1)\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} x^{\alpha+\gamma}$$

One expects a generalized Dirac delta function to give **(1G)** $\int \delta(\xi)(d\xi)^\alpha = \alpha x^{\alpha-1}$ and using (1.9) the fractional derivative of this delta function can be defined via **(1H)** $\int \delta^{(\alpha)}(\xi)f(\xi)(d\xi)^\alpha = -\int \delta(\xi)f^{(\alpha)}(\xi)(d\xi)^\alpha$ ($0 < \alpha \leq 1$) and (1.17) gives then **(1I)** $\int \delta^{(\alpha)}(\xi)f(\xi)(d\xi)^\alpha = -\alpha x^{\alpha-1}f^{(\alpha)}(0)$ ($0 < \alpha \leq 1$). The relation between fractional integral and derivative is

$$(1.22) \quad \frac{d^\alpha}{dx^\alpha} \int_0^x f(\xi)(d\xi)^\alpha = \Gamma(1 + \alpha)f(x) = \alpha!f(x);$$

$$\frac{d}{dx^\alpha} \int_0^{u(x)} f(\xi)(d\xi)^\alpha = \alpha!f(u(x))(u'(x))^\alpha$$

This leads to the useful result **(★)** $y = (1/\alpha!) \int_0^x y^{(\alpha)}(\xi)(d\xi)^\alpha$.

Now for coarse graining and fractal space it appears that α should be the box (or perhaps Hausdorff) dimension of the space determined via **(1J)** $d = \lim_{\epsilon \rightarrow 0^+} [-(\log N(\epsilon)/\log(\epsilon))]$ where $N(\epsilon)$ is the number of ϵ balls needed to cover the curve. Thus $N \sim (L/\epsilon^\alpha)$ and **(1K)** $d = \lim_{\epsilon \rightarrow 0} [-\log(\epsilon^\alpha/\log(\epsilon))]$. In this framework the velocity $\dot{x}_\alpha(t)$ is defined by **(1L)** $u_\alpha(t) = \dot{x}_\alpha(t) = (dx)^\alpha/dt$; $dx > 0$, $0 < \alpha < 1$. For coarse graining in both time and space we have ($0 < \beta < 1$, $(dt)^\beta > dt$)

$$(1.23) \quad v_\beta(t) = x^{(\beta)}(t) = \frac{d^\beta x}{(dt)^\beta} = \beta! \frac{dx}{(dt)^\beta}$$

This leads to a time-space coarse graining

$$(1.24) \quad \dot{x}_{\alpha,\beta}(t) = \frac{(dx)^\alpha}{(dt)^\beta} = \left[\frac{(dx)^{\alpha/\beta}}{dt} \right]^\beta = (\dot{x}_{\alpha/\beta})^\beta \quad (0 < \alpha < \beta < 1)$$

Some preliminary results are e.g.

- (1) Given $y = f(x)$ and $x = g(y)$ one has **(1M)** $y^{(\alpha)}(x)x^{(\alpha)}(y) = [(1 - \alpha)!]^{-2}(xy)^{1-\alpha}$. This follows from

$$(1.25) \quad y^{(\alpha)}(x)x^{(\alpha)}(y) = \left(\frac{d^\alpha y}{dx^\alpha} \right) \left(\frac{d^\alpha x}{dy^\alpha} \right) = \left(\frac{d^\alpha y}{dy^\alpha} \right) \left(\frac{d^\alpha x}{dx^\alpha} \right)$$

and **(1N)** $D^\alpha x^\gamma = \Gamma(\gamma + 1)\Gamma^{-1}(\gamma + 1 - \alpha)x^{\gamma-\alpha}$ which implies **(1O)** $(1 - \alpha)!d^\alpha x = x^{1-\alpha}(dx)^\alpha$. If now we assume that $x(t)$ is a

function of time which is discontinuous in such a manner that one can write $d^\alpha t = \alpha! dt$ then **(1L)** implies

$$(1.26) \quad \dot{x}_\alpha(t) = \frac{\alpha!(dx)^\alpha}{\alpha!dt} = \alpha! \frac{(dx)^\alpha}{d^\alpha t} = \alpha! [t^{(\alpha)}(x)]^{-1}$$

Putting **(1M)** into (1.26) yields

$$(1.27) \quad \dot{x}_\alpha(t) = \alpha!((1-\alpha)!)^2 (xt)^{\alpha-1} x^{(\alpha)}(t) = \rho(\alpha)(xt)^{\alpha-1} x^{(\alpha)}(t)$$

with $\rho(\alpha) = \alpha![(1-\alpha)!]^2$.

The Mittag-Leffler function is defined via **(1P)** $E_\alpha(t^\alpha) = \sum_{k=0}^{\infty} [t^{\alpha k} / (\alpha k)!]$ where $(\alpha k)! = \Gamma(1+\alpha k)$. It satisfies the equation **(1Q)** $D^\alpha E_\alpha(t^\alpha) = E_\alpha(t^\alpha)$ and further

$$(1.28) \quad D^\alpha K = 0 \quad (K = \text{constant}); \quad D^\alpha x^{\alpha k} = \frac{(\alpha k)!}{(\alpha k - \alpha)!} x^{\alpha(k-1)}$$

$$f(x+h) = E_\alpha[h^\alpha D_x^\alpha f(x)]$$

One can state in particular

PROPOSITION 1.1. Given two real functions $f, g : \mathbf{R} \rightarrow \mathbf{R}$ of the form $f \rightarrow f(u)$ and $x \rightarrow u(x)$ then

$$(1.29) \quad \frac{d^\alpha f(u(x))}{dx^\alpha} = \frac{d^\alpha f(u)}{du^\alpha} \left(\frac{du}{dx} \right)^\alpha;$$

$$f_x^{(\alpha)}(u(x)) = (1-\alpha)! u^{\alpha-1} f_u^{(\alpha)}(u) u_x^{(\alpha)}(x)$$

Proof: First note that (1.29A) follows in writing

$$(1.30) \quad \frac{d^\alpha f(u(x))}{dx^\alpha} = \frac{d^\alpha f(u)}{du^\alpha} \frac{(du)^\alpha}{(dx)^\alpha} = \frac{d^\alpha f(u)}{du^\alpha} (u')^\alpha$$

Then putting (1.12) into (1.30) gives

$$(1.31) \quad \frac{d^\alpha f}{dx^\alpha} = \frac{d^\alpha f(u)}{du^\alpha} (1-\alpha)! u^{\alpha-1} \frac{d^\alpha u}{dx^\alpha}$$

Note that if a function has a derivative then it has an α derivative for any α ($0 < \alpha < 1$). As an example note that for $f(x) = E_\alpha(\lambda \log^\alpha(x))$

$$(1.32) \quad \frac{d^\alpha f(x)}{dx^\alpha} = \frac{d^\alpha E_\alpha(\lambda \log^\alpha(x))}{(d(\log(x)))^\alpha} \left(\frac{d(\log(x))}{dx} \right)^\alpha = \lambda x^{-\alpha} E_\alpha(\lambda \log^\alpha(x))$$

Another example involves $y = y(x)$, $x = g(y)$, $y = f(g(y))$. Then

$$(1.33) \quad y^{(\alpha)}(x) x^{(\alpha)}(y) = \frac{d^\alpha y}{dx^\alpha} \frac{d^\alpha x}{(dy)^\alpha} = \frac{d^\alpha y}{dy^\alpha} \frac{d^\alpha x}{dx^\alpha}$$

Taking account of (1.9) there results

$$(1.34) \quad y^{(\alpha)}(x) x^{(\alpha)}(y) = (1-\alpha)!^{-2} (xy)^{1-\alpha}$$

As a variation consider $y = E_\alpha(x)$ and $x^\alpha = \text{Log}_\alpha y$ where **(1R)** $y^\alpha(x) = E_\alpha(x^\alpha)$ and (1.34) yields then

$$(1.35) \quad E_\alpha(x^\alpha)x^{(\alpha)}(y) = ((1-\alpha)!)^{-2}(xy)^{1-\alpha}$$

Therefore

$$(1.36) \quad x^{(\alpha)}(y) = D^\alpha[(\text{Log}_\alpha(y))^{1/\alpha}] = \frac{1}{((1-\alpha)!)^2} \frac{(xy)^{1-\alpha}}{E_\alpha(x^\alpha)}$$

Consequently

$$(1.37) \quad D^\alpha[\text{Log}_\alpha(y)]^{1/\alpha} = \frac{1}{((1-\alpha)!)^2} \frac{y^{1-\alpha}}{y} [(\text{Log}_\alpha(y))^{\frac{1-\alpha}{\alpha}}]$$

A function $x(t)$ is said to be self-similar with Hurst exponent $H > 0$ whenever **(1S)** $x(at) \propto a^H x(t) \Rightarrow x(t) \simeq t^H x(1)$. (1.18) implies also **(1T)** $x(t) - x(0) \simeq bt^\alpha$ for small t and therefore $x(at) - x(0) \simeq a^\alpha(x(t) - x(0))$. This means that if $x(t)$ is α differentiable at $t = 0$ then $x(t)x(0)$ is locally self similar with Hurst exponent α and there is also a kind of converse. In any event self similarity and Hurst exponents are intimately connected with fractional calculus and since self similarity and Hölder exponents are characteristic of fractals we can see how the fractal-fractional connection arises (cf. also [8]). We add now a few formulas from [43] (especially from papers in 2009 and 2012 - with possible repetitions from above):

$$(1.38) \quad \int \frac{d^\alpha x}{x} = \text{Log}_\alpha\left(\frac{x}{C}\right); \quad x = E_\alpha(\text{Log}_\alpha(x)); \quad [x/C] > 0$$

Further

- (1) $\text{Log}_\alpha(x^y) = y^\alpha \text{Log}_\alpha(x)$; $E_\alpha(x^\alpha y^\alpha) = E_\alpha(y^\alpha)^x$
- (2) $(\text{Log}_\alpha(uv))^{1/\alpha} = (\text{Log}_\alpha u)^{1/\alpha} + (\text{Log}_\alpha v)^{1/\alpha}$; $(dx)^\alpha = (\alpha!)^{-1} d^\alpha(x^\alpha)$
- (3) $E_\alpha(\lambda(x+y)^\alpha) = E_\alpha(\lambda x^\alpha) + E_\alpha(\lambda y^\alpha)$; $\alpha!(1-\alpha)!dx = x^{1-\alpha}(dx)^\alpha$
- (4) For MRL but not RL one has

$$D^\alpha E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha); \quad D^\alpha E_\alpha(u^\alpha(x)) = E_\alpha(u^{(\alpha)})(u'(x)^\alpha)$$

Another collection of formulas from [43] (APL, 22 (2009), 378-385) is given (with possible repetitions) via $(\Gamma_\alpha(n+1) = (\alpha!)^n n!)$

(1.39)

$$D^\alpha E_\alpha(\lambda x) = \lambda \alpha^{-\alpha} x^{1-\alpha} E_\alpha(\lambda x); \quad \Gamma_\alpha(x) = (\alpha!)^{-1} \int_0^\infty E_\alpha(-t^\alpha) t^{\alpha-1} (dt)^\alpha;$$

$$\int (dx)^\alpha = \Gamma^{-1}(2-\alpha) \int x^{1-\alpha} (dx)^\alpha; \quad \Gamma_\alpha(x+1) = (\alpha!) x \Gamma_\alpha(x);$$

$$\frac{d^\alpha \text{Log}_\alpha(x)}{dx^\alpha} = \frac{\alpha!}{(1-\alpha)!} \frac{1}{x^\alpha}; \quad B_\alpha(x, y) = \int_0^1 (1-t)^{(x-1)\alpha} t^{(y-1)\alpha} (dt)^\alpha$$

$$\begin{aligned}
\int f^{(\alpha)}(x)(dx)^\alpha &= \int d^\alpha f = \alpha! f(x); \quad B_\alpha(x, y) = \frac{\Gamma_\alpha(x)\Gamma_\alpha(y)}{\Gamma_\alpha(x+y)} \\
Log_\alpha(x) &= \Gamma^{-1}(2-\alpha) \int_0^x \left(\frac{d\xi}{\xi}\right)^\alpha; \quad D^\alpha g(u(x), x) = g'_u u^{(\alpha)}(x) + \Gamma^{-2}(2-\alpha) x^{1-\alpha} g'_x \\
x &= E_\alpha \left(\Gamma^{-1}(2-\alpha) \int_0^x \left(\frac{d\xi}{\xi}\right)^\alpha \right); \quad \int \frac{(dx)^\alpha}{x} = \frac{x^{\alpha-1}}{(\alpha-1)\Gamma(2-\alpha)}; \\
\int_{\mathbf{R}} E_\alpha \left(- \left(\frac{x^2}{2\sigma^2} \right)^\alpha \right) (dx)^\alpha &= (2\sigma)^\alpha \sqrt{\frac{(\alpha!)^3}{(2\alpha)!}} \pi^{\sigma/2}
\end{aligned}$$

2. FRACTIONAL SCHRÖDINGER EQUATION

We go here to [43] (CSF, 41 (2009), 1590-1604) and write first some useful formulas:

$$(2.1) \quad v_\beta(t) = x^{(\beta)}(t) = \frac{d^\beta x}{(dt)^\beta} = \beta! \frac{dx}{(dt)^\beta}$$

$$(2.2) \quad y^{(\alpha)}(x) x^{(\alpha)}(y) = \left(\frac{d^\alpha y}{dx^\alpha} \right) \left(\frac{d^\alpha x}{dy^\alpha} \right) = \left(\frac{d^\alpha y}{dy^\alpha} \right) \left(\frac{d^\alpha x}{dx^\alpha} \right)$$

$$(2.3) \quad u_\alpha = \dot{x}_\alpha(t) = \frac{\alpha! (dx)^\alpha}{\alpha! dt} = \alpha! \frac{(dx)^\alpha}{d^\alpha t} = \alpha! [t^{(\alpha)}(x)]^{-1}$$

From (1M) and (2.3) we get then

$$(2.4) \quad u_\alpha(t) = \alpha! ((1-\alpha)!)^2 (xt)^{\alpha-1} x^{(\alpha)}(t) = \rho(\alpha) (xt)^{\alpha-1} x^{(\alpha)}(t)$$

For the Schrödinger equation (SE) we can look at a Lagrangian and action ($q \sim x$)

$$(2.5) \quad S = \int_{t_1}^{t_2} L(q, u_\alpha, \tau) (d\tau)^\alpha; \quad L = K(q, u_\alpha, t) - V(q); \quad K = \frac{1}{2} \sum_{ij} (u_\alpha)_i (u_\alpha)_j$$

where $V(q)$ is a potential energy term. Then (assuming suitable differentiability)

$$(2.6) \quad \delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial u_\alpha} \frac{\partial u_\alpha}{\partial q} \delta q + \frac{\partial L}{\partial u_\alpha} \frac{\partial u_\alpha}{\partial q^\alpha} (\delta q)^\alpha \right] (dt)^\alpha$$

In order to convert the integral involving $(\delta q)^\alpha$ to an integral in terms of δq one uses the formula

$$(2.7) \quad \int (fg)^{(\alpha)} (dt)^\alpha = \int f^{(\alpha)} (dt)^\alpha + \int f g^{(\alpha)} (dt)^\alpha$$

leading to the dynamical equation

$$(2.8) \quad \frac{\partial L}{\partial q} - \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial L}{\partial u_\alpha} \frac{\partial u_\alpha}{\partial q^{(\alpha)}} \right) + F = 0; \quad \frac{\partial L}{\partial q} - \frac{d^\alpha}{dt^\alpha} \left(\frac{\partial L}{\partial q^{(\alpha)}} \right) + F = 0$$

where F denotes an external applied force. Then write in Hamiltonian form

$$(2.9) \quad H \sum p_i q_i^{(\alpha)} - L(q, u_\alpha(q, q^{(\alpha)}, t), t); \quad q^{(\alpha)}(t) = \frac{\partial H}{\partial p}; \quad p^{(\alpha)}(t) = -\frac{\partial H}{\partial q}$$

which are strictly equivalent to (2.8).

Now a probability density of fractional order is introduced as follows. Let X be a random variable on $[a, b]$ and $p_\alpha(x)$ be a positive function on $[a, b]$. Then X is referred to as a random variable of fractional order α , $0 < \alpha < 1$ with probability $p_\alpha(x)$ whenever

$$(2.10) \quad F(x', x) = Pr(x' < X < x) = \frac{1}{\Gamma(1 + \alpha)} \int_{x'}^x p_\alpha(\xi)(d\xi)^\alpha; \quad F(a, b) = 1$$

According to (2.10) and

$$(2.11) \quad dy = (f(x)(dx)^\alpha) \Rightarrow y = \int_0^x f(\xi)(d\xi)^\alpha = \alpha \int_0^x (x - \xi)^{\alpha-1} f(\xi) d\xi; \quad (0 < \alpha \leq 1)$$

$$\Rightarrow \frac{d}{dx^\alpha} \int_0^x f(\xi)(d\xi)^\alpha = \alpha! f(x)$$

if we denote by $p(x)$ the corresponding probability density of X we should have the identity

$$(2.12) \quad p(x) = \Gamma^{-1}(\alpha)(a - x)^{\alpha-1} p_\alpha(x), \quad (x \leq a)$$

In other words the fractional probability density $p_\alpha(x)$ can be thought of as a family of standard probability density functions $p(x)$. Then using (2.10) there results

$$(2.13) \quad \frac{\partial^\alpha F(x', x)}{\partial x^\alpha} = p_\alpha(x)$$

Now the Hamiltonian function associated with a particle in a force field determined by the potential function $V(x, t)$ is **(2A)** $H = (1/2m)p^2 + V(x, t)$ with $p = mv$ and for the SE one takes **(2B)** $p \rightarrow -i\hbar\nabla$, $p^2 \rightarrow -\hbar^2\Delta$ so that **(2C)** $i\hbar\psi_t = -(\hbar^2/2m)\psi_{xx} + V(x, t)$. Classically

$$(2.14) \quad E = (1/2m)p^2 + V, \quad E \rightarrow -i\hbar\partial_t, \quad i\hbar\psi_t = -(\hbar^2/2m)\psi_{xx} + V\psi$$

Now write the energy as **(2D)** $E = (1/2m)\rho^2(\alpha)(xt)^{2(\alpha-1)}p_\alpha^2$, based on $p_\alpha \sim mx^{(\alpha)} \sim -i\hbar\nabla$, with velocity **(2E)** $u_\alpha = (dx)^\alpha/dt = \rho(\alpha)(xt)^{\alpha-1}x^{(\alpha)}$ as in (2.4). Then with $E \rightarrow i\hbar\partial_t^{(\alpha)}$ one obtains the SE

$$(2.15) \quad i\hbar\psi_t^{(\alpha)} = -\frac{\hbar^2}{2m}\rho^2(\alpha)(xt)^{2(\alpha-1)}\psi_{xx} + V\psi$$

This last step is somewhat heuristic and it is justified in [43] by constructing an example where it can hold. We would prefer however to consider

Schödinger equations written directly in terms of fractional derivatives, e.g.

$$(2.16) \quad i\hbar D_t^\beta \psi = -\frac{\hbar^2}{2m} D_x^{2\alpha} \psi + V\psi$$

which facilitate the introduction of quantum potentials (QP).

In the case of (2.15) one could imagine a QP of the form

$$(2.17) \quad \hat{\Omega} = -\frac{\hbar^2}{2m} \rho^2(\alpha)(xt)^{2(\alpha-1)} \frac{R_{xx}}{R}$$

and the entrance of time here and in (2.15) suggests perhaps memory effects in the fractional context (cf. [37, 51, 74]). In [14] we worked with a fractional SE (**2F**) $i\hbar \partial_t \psi = (\hbar^2/2m) d_\alpha^2 \psi + V(x)\psi$ where d_α was a very special fractional derivative (**2G**) $d_\alpha z^\beta = \beta z^{\beta-\alpha}$ and a space of Puiseux functions was used (cf. [14, 48]). A function \tilde{E}_α (resembling E_α in its properties) was used with a wave function $\psi = R\tilde{E}_\alpha(iS/\hbar)$ to solve (**2F**) and determine a quantum potential (QP) (cf. Remark 2.1 below)

$$(2.18) \quad \Omega_\alpha = -\frac{\hbar^2 d_\alpha^2 R}{2mR}$$

Here it might be possible to find a similar construction in the MRL theory and we note in passing that a possibly macro framework for the SE and QP arises in the Nottale theory ([59, 60]). The time variable will be assumed classical here and $D^{2\alpha} = D^\alpha D^\alpha$ for simplicity ((R, S) depend on (x, t)).

REMARK 2.1. The function \tilde{E}_α of [48] can be written as

$$(2.19) \quad \tilde{E}_\alpha(z) = \sum_0^\infty \frac{(z^\alpha/\alpha)^k}{\Gamma(k+1)} = \exp\left(\frac{z^\alpha}{\alpha}\right)$$

This does not seem to be in the general Mittag-Leffler class (cf. [20, 37, 43, 45]) however. In any event in [14] we used (**2H**) $\tilde{E}_\alpha = \sum_0^\infty (z^\alpha/\alpha)^k / \Gamma(k+1) \sim \exp[z^\alpha/\alpha]$ so that

$$(2.20) \quad d_\alpha \left(\frac{z^\alpha}{\alpha}\right)^k = d_\alpha \left(\frac{z^{\alpha k}}{\alpha^k}\right) = \alpha^{-k} \alpha k z^{\alpha k - \alpha} = \alpha^{1-k} k z^{\alpha(k-1)}$$

This means that

$$(2.21) \quad d_\alpha \tilde{E}_\alpha(z) = \sum_1^\infty \frac{\alpha^{1-k}}{(k-1)!} z^{\alpha(k-1)} = \sum_0^\infty \left(\frac{z}{\alpha}\right)^n \frac{1}{n!} = \tilde{E}_\alpha(z)$$

We note here an oversight in [14] (math-ph 1206.0900) where we used an analogue of E_α , i.e. $\tilde{E}_\alpha(z) = \exp[z^\alpha/\alpha] = \sum_0^\infty (z^\alpha/\alpha)^k / k!$. With the Kobelev derivative in [48] given by $d_\alpha z^\gamma = \gamma z^{\gamma-\alpha}$ one has $d_\alpha \tilde{E}_\alpha = \tilde{E}_\alpha$ and there is a chain rule for the d_α calculus. In [14] we wrote $\tilde{E}_\alpha(i\hat{S})$ for solution of a Schrödinger equation (= SE) (**2I**) $i\hbar \psi_t = -[(\hbar^2)/(2m)]\Delta\psi + (V/2m)\psi$

and overlooked some features ($\hat{S} = S/\hbar$ is used). In particular in using \tilde{E}_α we should have emphasized $i\hat{S} \sim z^\alpha/\alpha$ and the calculations are then somewhat different; however the result is again (2.18). Thus if we take $z^\alpha/\alpha = i\hat{S}$ with $S = S(x, t)$ there results from [48] **(2J)** $d_\alpha^z \tilde{E}(z) = \tilde{E}(z)$ and e.g. **(2K)** $\partial_t \tilde{E}_\alpha(z) = \tilde{E}_\alpha(z) \partial_t z$ where

$$(2.22) \quad i\hat{S}_t = \frac{1}{\alpha} \alpha z^{\alpha-1} \partial_t z \Rightarrow \frac{\hat{S}_t}{\hat{S}} = \frac{z_t}{z}$$

Also

$$(2.23) \quad id_\alpha^x \hat{S} = z^{\alpha-1} d_\alpha^x z = z^{\alpha-1} z_\alpha^x;$$

$$id_{2\alpha}^x \hat{S} = d_\alpha^x (z^{\alpha-1} z_\alpha^x) = (\alpha-1) z^{\alpha-2} (z_\alpha^x)^2 + z^{\alpha-1} z_{2\alpha}^x$$

By analogy with the classical SE with $d_\alpha^x \sim \partial_x$ we can posit

$$(2.24) \quad i\hbar(R_t \tilde{E}_\alpha + R \tilde{E}_\alpha z_t) = -\frac{\hbar^2}{2m} d_\alpha^x [d_\alpha^x R \tilde{E}_\alpha + R \tilde{E}_\alpha z_\alpha^x] + V R \tilde{E}_\alpha =$$

$$= \frac{\hbar^2}{2m} \left[(d_{2\alpha}^x R) \tilde{E}_\alpha + 2d_\alpha^x R z_\alpha^x \tilde{E}_\alpha + R (z_\alpha^x)^2 \tilde{E}_\alpha + R \tilde{E}_\alpha z_{2\alpha}^x \right] + V R \tilde{E}_\alpha$$

Then canceling the \tilde{E}_α terms in (2.31) gives

$$(2.25) \quad i\hbar(R_t + R z_t) = -\frac{\hbar^2}{2m} [d_{2\alpha}^x R + 2d_\alpha^x R z_\alpha^x + R (z_\alpha^x)^2 + R z_{2\alpha}^x] + V R$$

Using (2.23) now we have

$$(2.26) \quad z_\alpha^x = iz^{1-\alpha} d_\alpha^x \hat{S}; \quad z^\alpha = i\alpha \hat{S}; \quad z^{\alpha-1} z_{2\alpha}^x = d_{2\alpha}^x \hat{S} -$$

$$-(\alpha-1) z^{\alpha-2} \left[-z^{2(1-\alpha)} (d_\alpha^x \hat{S})^2 \right] = id_{2\alpha}^x \hat{S} + i \frac{(1-\alpha)}{\alpha} \frac{(d_\alpha^x \hat{S})^2}{\hat{S}}$$

Thus we can use

$$(2.27) \quad z_t = (\hat{S}_t/\hat{S})z; \quad d_\alpha^x z = z_\alpha^x = iz^{1-\alpha} d_\alpha^x \hat{S}; \quad z^\alpha = i\alpha \hat{S}$$

$$z^{(\alpha-1)} z_{2\alpha}^x = id_{2\alpha}^x \hat{S} + i \frac{(1-\alpha)}{\alpha} \frac{(d_\alpha^x \hat{S})^2}{\hat{S}}$$

(note that $z^{-\alpha} = \alpha/[\alpha z^\alpha] = i/\alpha \hat{S}$).

Now to determine how to split (2.25) we have to look at the complex numbers involved. Thus $z^\alpha/\alpha = iS/\hbar$ and $z = i\hat{S}\alpha$. We assume \hat{S} and α are real with z complex so that **(2L)** $z = [\hat{S}\alpha]^{1/\alpha} i^{1/\alpha} = [\hat{S}\alpha]^{1/\alpha} [\exp(i\pi/2\alpha)]$. In this event we consider first the ingredients in (2.27) as complex numbers via the related z terms. The terms in d_α^x , $d_{2\alpha}^x S, \dots$ are all real so one can most easily determine the form of the z_α^x , $z_{2\alpha}^x$, etc. from (2.22)-(2.27).

- (1) z_t/z is real
- (2) $z^{\alpha-1} z_\alpha^x$ is imaginary

- (3) $z^{\alpha-2}(z_\alpha^x)^2 = (1/z)(z^{\alpha-1}z_\alpha^x)z_\alpha^x$ is real (as the product of two imaginaries)
- (4) $z^{\alpha-1}z_{2\alpha}^x$ is imaginary
- (5) z^α is imaginary

Now refer to the classical situation where

$$(2.28) \quad \hat{S}_t + \frac{\hat{S}_x^2}{2m} + V - \frac{\hbar^2 R''}{2mR} = 0; \quad \partial_t R + \frac{1}{m}(2R'\hat{S}' + R\hat{S}'') = 0$$

and the quantum potential (QP) is **(2M)** $\mathfrak{Q} = -[\hbar^2 R''/2mR]$. Thus the correct choice from (2.24)-(2.25) appears to be

$$(2.29) \quad iR_t = -\frac{1}{2m}[2d_\alpha^x R z_\alpha^x + R z_{2\alpha}^x]; \quad iR z_t = \frac{1}{2m}[(d_{2\alpha}^x R + R(z_\alpha^x)^2 + VR]$$

and we suggest a fractional QP

$$(2.30) \quad \mathfrak{Q}_\alpha = -\frac{\hbar^2 d_{2\alpha}^x R}{2mR}$$

which, not too surprisingly, agrees with [14]. ■

Now the SE (2.15) of Jummarie was based on D_x^α but conceals the information when expressed as in (2.15) in terms of ∂_t and ∂_x . In particular it makes it clumsy to identify a QP. Hence we will check now a SE with D^α for MRL of the type in (2.16) which immediately produces a QP (cf. also [37] for SE of type (2.16)). Thus for $\psi = R \exp(iS/\hbar)$ we write as usual

$$(2.31) \quad D_x^\alpha f(x) = f^{(\alpha)} = \frac{d^\alpha f}{(dx)^\alpha}$$

with the notation of Section 1. Then consider a SE of classical form

$$(2.32) \quad i\hbar \partial_t \psi = -\frac{\hbar^2}{2m}(D_x^{2\alpha} \psi) + V\psi$$

for $\psi = R \exp(iS/\hbar)$ which gives rise to the standard equations for $D_x^{2\alpha} \rightarrow \Delta$ (cf. [13]). Recall that D^α satisfies a chain rule and we look first for coarse grained space and classical time in writing

$$(2.33) \quad \begin{aligned} D^\alpha [R e^{iS/\hbar}] &= D^\alpha R e^{iS/\hbar} + \frac{i}{\hbar} D^\alpha R D^\alpha S e^{iS/\hbar}; \\ D^{2\alpha} [R e^{iS/\hbar}] &= D^{2\alpha} R e^{iS/\hbar} + \frac{2i}{\hbar} D^\alpha R e^{iS/\hbar} + \\ &\quad + R \left(\frac{i}{\hbar}\right)^2 \frac{D^{2\alpha} S}{\hbar} e^{iS/\hbar} + R \left(\frac{i D^\alpha S}{\hbar}\right)^2 e^{iS/\hbar} \end{aligned}$$

leading to (via (2.32))

$$(2.34) \quad i\hbar \left(\partial_t R + \frac{i R S_t}{\hbar} \right) = -\frac{\hbar^2}{2m} \left[D^{2\alpha} R + \frac{2i D^\alpha R D^\alpha S}{\hbar} - \frac{R}{\hbar^2} D^{2\alpha} S - \frac{R (D^\alpha S)^2}{\hbar^2} \right]$$

which becomes

$$(2.35) \quad i\hbar R_t - RS_t = -\frac{\hbar^2}{2m}D^{2\alpha}R - \frac{i\hbar}{m}D^\alpha R D^\alpha S - \frac{R}{2m}D^{2\alpha}S - \frac{R}{2m}(D^\alpha S)^2 + \frac{VR}{m}$$

and finally

$$(2.36) \quad \partial_t R^2 = \frac{1}{m}D^\alpha(R^2)D^\alpha S; \quad -RS_t = -\frac{\hbar^2}{2m}D^{2\alpha}R - \frac{1}{2m}(D^\alpha S)^2 - \frac{V}{m}$$

with QP

$$(2.37) \quad \Omega^\alpha = -\frac{\hbar^2 D^{2\alpha} R}{2mR}$$

as in (2.30). Evidently this form will persist for any fractional derivative with a chain rule and Leibnitz rule for products. If in addition t is coarse grained with a D^β derivative then the $i\hbar\partial_t\psi$ term in (2.37) becomes

$$(2.38) \quad i\hbar D_t^\beta (Re^{iS/\hbar}) = i\hbar \left[(D^\beta R)e^{iS/\hbar} + R \frac{i}{\hbar} D_t^\beta S e^{iS/\hbar} \right]$$

so in (2.36B) we can replace ∂_t by D^β and the QP is unaffected. Indeed (2.38) becomes

$$(2.39) \quad D_t^\beta R^2 = \frac{1}{m}D^\alpha(R^2)D^\alpha S; \quad -RD_t^\beta S = -\frac{\hbar^2}{2m}D^{2\alpha}R - \frac{1}{2m}(D^\alpha S)^2 - \frac{V}{m}$$

Such results will hold for any choice of fractional derivatives satisfying the chain rule and Leibnitz product rules. It is worth noting that the MRL derivatives are non-local and solutions (if any) of fractional SE may be sparse or have rather different properties than in the traditional situations (cf. [4, 41] for example).

3. Q-DEFORMED AND FRACTIONAL CALCULUS

We extract here from [37] (1007.1084) and refer also to other citations in [37] for more information (especially [6]). It can be shown that the concept of q-deformed Lie algebras and the methods developed in fractional calculus are closely related and may be combined leading to a new class of fractional q-deformed Lie algebras (cf. also [12] for q-calculus). In order now to describe a deformed Lie algebra we introduce a parameter q and define a mapping

$$(3.1) \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}; \quad \lim_{x \rightarrow 1} [x]_q = x; \quad [0]_q = 0$$

$$(3.2) \quad D_x^q f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}; \quad D_x^q x^n = [n]_q x^{n-1}$$

As an example of a q -deformed Lie algebra one looks at the harmonic oscillator. The creation and annihilation operators a^\dagger , a and the number operator n generate the algebra

$$(3.3) \quad [N, a^\dagger] = a^\dagger; [N, a] = -a; aa^\dagger - q^{\pm 1}a^\dagger a = q^{\mp N}$$

Via (3.1) and alternative definition for (3.3) is given via (\bullet) $a^\dagger a = [N]_q$ with $aa^\dagger = [N+1]_q$. One defines a vacuum state with $a|0\rangle = 0$ and the action of the operators $\{a, a^\dagger, N\}$ on the basis $|n\rangle$ of a Fock space is given by

$$(3.4) \quad N|n\rangle = n|n\rangle; a^\dagger|n\rangle = \sqrt{[n+1]_q}|n+1\rangle; a|n\rangle = \sqrt{[n]_q}|n-1\rangle$$

The Hamiltonian of the q -deformed harmonic oscillator and its eigenvectors are defined via

$$(3.5) \quad H = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a); E^q(n) = \frac{\hbar\omega}{2}([n]_q + [n+1]_q)$$

In [37] various fractional derivatives are recalled (cf. Sections 1-2) and in particular one mentions the Caputo derivative

$$(3.6) \quad D_x^\alpha = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^x dx (x-\xi)^{-\alpha} \partial_\xi f(\xi) & 0 \leq \alpha < 1 \\ \frac{1}{\Gamma(2-\alpha)} \int_0^x d\xi (x-\xi)^{1-\alpha} (\partial^2 f(\xi)/\partial \xi^2) & 1 \leq \alpha < 2 \end{cases}$$

Then for $x^{n\alpha}$

$$(3.7) \quad D_x^\alpha x^{n\alpha} = \begin{cases} \frac{\Gamma(1+n\alpha)x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)} & n > 0 \\ 0 & n = 0 \end{cases}$$

The fractional derivative parameter α can be interpreted as a deformation parameter via $|n\rangle = x^{n\alpha}$ and

$$(3.8) \quad [n]_\alpha |n\rangle = \begin{cases} \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} |n\rangle & n > 0 \\ 0 & n = 0 \end{cases}; \lim_{\alpha \rightarrow 1} [n]_\alpha = n$$

Then via (3.2) the standard q -numbers can be defined more or less heuristically and there are different possibilities. On the other hand the q -deformation based on a fractional calculus α is uniquely determined once a set of basis vectors is given and the harmonic oscillator will be used as an illustration. This means that information about q -entropy or q -information can be connected to any background α -fractional derivatives.

Now replace x and p by \hat{x} and \hat{p} to get into QM and write

$$(3.9) \quad \hat{X}f(x) = xf(x); \hat{P}f(x) = -i\hbar\partial_x f(x); [\hat{X}, \hat{P}] = i\hbar$$

Using D_x^α there follows now

$$(3.10) \quad \hat{x} = \left(\frac{\hbar}{mc}\right)^{1-\alpha} x^\alpha; \hat{p} = -i \left(\frac{\hbar}{mc}\right)^\alpha mc D_x^\alpha$$

The classical and quantum Hamiltonians are now

$$(3.11) \quad H_{class} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2; \quad H^\alpha = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$$

and the SE becomes

$$(3.12) \quad H^\alpha \psi = \left[-\frac{1}{2m} \left(\frac{\hbar}{mc} \right)^2 m^2 c^2 D_x^\alpha D_x^\alpha + \frac{1}{2} m \omega^2 \left(\frac{\hbar}{mc} \right)^{2(1-\alpha)} x^{2\alpha} \right] \psi = E \psi$$

The ‘‘Hermiticity’’ of such an operator will depend of course on the choice of fractional derivative and it can be shown that the Feller and Riesz fractional derivatives (but not Caputo or Riemann-Liouville) will insure Hermiticity (cf. [51, 37]). Here the Riesz derivative is

$$(3.13) \quad D_R^\alpha f(x) = \Gamma(1+\alpha) \frac{\sin(\pi\alpha/2)}{\pi} \int_0^\infty \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{\alpha+1}} d\xi; \quad (0 < \alpha < 2)$$

and the Feller derivative is

$$(3.14) \quad {}_F D_1^\alpha f(x) = \Gamma(1+\alpha) \frac{\cos(\pi\alpha/2)}{\pi} \int_0^\infty \frac{f(x+\xi) - f(x-\xi)}{\xi^{\alpha+1}} d\xi; \quad (0 \leq \alpha < 1)$$

For a canonical picture a scaled energy E^q and coordinates can be introduced via

$$(3.15) \quad \xi^\alpha = \sqrt{\frac{m\omega}{\hbar}} \left(\frac{\hbar}{mc} \right)^{1-\alpha} x^\alpha; \quad E = \hbar\omega E^\alpha$$

leading to the Hamiltonian

$$(3.16) \quad H^\alpha \psi_n(\xi) = \frac{1}{2} [-D_\xi^{2\alpha} + \xi^{2\alpha}] \psi_n(\xi) = E'(n, \alpha) \psi_n(\xi)$$

Laskin (cf. [51]) has derived an approximate analytic solution within the framework of the WKB approximation which has the advantage of being independent of the choice of a specific definition of the fractional derivatives (cf. [37, 51] for more information on this) and the result is

$$(3.17) \quad E'(n, \alpha) = \left[\frac{1}{2} + n \right]^\alpha \pi^{\alpha/2} \left[\frac{\alpha \Gamma(\frac{1+\alpha}{2\alpha})}{\Gamma(1/2\alpha)} \right]; \quad n = 0, 1, 2, \dots$$

(cf. also [33, 51, 74, 80]). We write also for a harmonic oscillator (following [37]) the connection to q-deformation arises from $n \gtrsim x^{n\alpha}$ with (♦) $E^q(n) = (\hbar\omega/2)([n]_q + [n+1]_q)$ while

$$(3.18) \quad [n]_\alpha |n\rangle = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} |n\rangle; \quad (\alpha \sim q)$$

for $n > 0$ (as in (3.7)-(3.8)). The q-deformation is then uniquely defined by α .

4. THE Q EXPONENTIAL FAMILY

We now begin to mix TD with the q-theory and from Naudts [58] (cond-mat 0911.5392) we recall first the Boltzmann-Gibbs distribution

$$(4.1) \quad f_\beta(x) = \frac{c(x)}{Z(\beta)} e^{-\beta H(x)}; \quad Z(\beta) = \int dx c(x) e^{-\beta H(x)}$$

Writing $\Phi(\beta) = \log(Z(\beta))$ for the Massieu function there results

$$(4.2) \quad \frac{d\Phi}{d\beta} = -U; \quad U = E_\beta H = \int dx f_\beta(x) H(x); \quad \frac{d^2\Phi}{d\beta^2} = E_\beta(H - U)^2 \geq 0$$

Thus $\Phi(x)$ is convex as expected from TD and in fact Φ is by definition the Legendre transform of the TD entropy $S(U)$ which is a convex function of U . The most common entropy functional is that of Boltzmann-Gibbs-Shannon

$$(4.3) \quad I(f) = - \int dx f(x) \log \frac{f(x)}{c(x)}$$

One knows that $I(f)$ takes its maximum at $f = f_\beta$ over the set of all f with $E_f H = U$. This maximal value of $I(f)$ is then identified with the TD entropy $S(U)$ via

$$(4.4) \quad S(U) = \max\{I(f); E_f H = U\} = I(f_\beta) = \\ = \int dx f_\beta(x) [\log(Z(\beta)) + \beta H(x)] = \log[Z(\beta)] + \beta U$$

Consequently

$$(4.5) \quad \Phi(\beta) = \max_U \{S(U) - \beta U\} \Rightarrow S(U) = \inf_\beta [\Phi(\beta) + \beta U] \Rightarrow \beta = \frac{dS}{dU}$$

Another consequence of (4.5) is that $S(U)$ is convex.

The Boltzmann-Gibbs distribution defines a statistical model with parameter β and it belongs to the so called exponential family. In fact a model belongs to the exponential family with parameters $\theta = \{\theta_1, \dots, \theta_n\}$ when its pdf $f_\theta(x)$ can be written in the canonical form

$$(4.6) \quad f_\theta(x) = c(x) \exp \left[\sum_1^n \theta_j K_j(x) - \alpha(\theta) \right]$$

Models of this family share a number of properties. For example they share identities generalizing (4.2)

$$(4.7) \quad \frac{\partial \alpha}{\partial \theta_j} = E_\theta K_j = \int dx f_\theta(x) K_j(x)$$

(the K_j may not depend on θ). We recall now the q -deformed logarithm and exponential

$$(4.8) \quad \log_q(u) = \frac{1}{1-q}(u^{q-1} - 1), \quad (u > 0); \quad \frac{d}{du} \log_q(u) = \frac{1}{u^q} > 0$$

The inverse function is

$$(4.9) \quad \exp_q(u) = [1 + (1-q)u]_+^{1/(1-q)}$$

Then **(4A)** $\exp_q[\log_q(u)] = u$; ($u > 0$). Further $\exp_q(0) = 1$ and

$$(4.10) \quad \frac{d}{du} \exp_q(u) = [\exp_q(u)]^q \geq 0$$

Now the notion of exponential family can be generalized by saying that the 1-parameter model $f_\theta(x)$ belongs to the q -exponential family if there exist functions $c(x)$, $\alpha(\theta)$, $H(x)$ such that **(4B)** $f_\theta(x) = c(x) \exp_q[-\alpha(\theta) - \theta H(x)]$. In the limit $q = 1$ this is the standard definition with $\theta = \beta$. Then introduce the q -deformed entropy functional

$$(4.11) \quad I_q(f) = - \int dx f(x) \log_q \left(\frac{f(x)}{c(x)} \right) = \frac{1}{1-q} \left[1 - \int dx c(x) \left(\frac{f(x)}{c(x)} \right)^{2-q} \right]$$

Assume now that $0 < q < 2$ and one can show that the pdf $f = f_\theta(x)$ maximizes the quantity

$$(4.12) \quad \frac{1}{2-q} I_q(f) - \theta E_f H$$

For $q = 1$ this is known as a variational principle (cf. [58]) which implies the maximum entropy principle $I_q(f_\theta) \geq I_q(f)$ for all f which satisfy $E_f H = E_\theta H$ (using the notation $E_\theta = E_{f_\theta}$). A well-known problem is now that f_θ also maximizes $\xi(I_q(f))$ where $\xi(u)$ is an arbitrary monotonically increasing function. Therefore a meaningful Ansatz is to assume that the TD entropy $S(U)$ is given via

$$(4.13) \quad S(U) = \xi(I_q(f_\theta)) = \xi(\alpha(\theta) + \theta E_\phi H)$$

The TD expression for the inverse temperature β can be then calculated using (4.5). Evidently the resulting relation between energy U and temperature β^{-1} will depend on the choice of the monotonic function $\xi(u)$.

The q -Gaussian distribution in one variable is

$$(4.14) \quad f(x) = \frac{1}{c_q \sigma} \exp_q(-x^2/\sigma^2)$$

with

$$(4.15) \quad c_q = \int_{-\infty}^{\infty} dx \exp_q(-x^2) = \sqrt{\frac{\pi}{q-1}} \frac{\Gamma(-\frac{1}{2} + \frac{1}{q-1})}{\Gamma(\frac{1}{q-1})} \quad (1 < q < 3)$$

$$= \sqrt{\frac{\pi}{1-q}} \frac{\Gamma\left(1 + \frac{1}{1-q}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{1-q}\right)} (q < 1)$$

It can be brought into the form (4B) with $c(x) = 1/c_q$, $H(x) = x^2$, $\theta = \sigma^{3-q}$ and

$$(4.16) \quad \alpha(\theta) = \frac{\sigma^{q-1}}{q-1} = \log_{2-q}(\sigma)$$

The $q = 1$ case reproduces the congenital Gauss distribution and for $q < 1$ the distribution vanishes outside an interval. Take for instance $q = 1/2$ in which case (4.14) becomes

$$(4.17) \quad f(x) = \frac{15\sqrt{2}}{32\sigma} \left[1 - \frac{x^2}{\sigma^2}\right]_+$$

This distribution vanishes outside the interval $[-\sigma, \sigma]$ and in the range $q < 3$ the q-Gaussian is strictly positive on the whole line and decays with a power law in $|x|$ instead of exponentially. For $q = 2$ there results (4C) $f(x) = (1/\pi)(\sigma/(x^2 + \sigma^2))$.

Apparently there is no consensus about the correct definition of the TD entropy $S(U)$ for isolated systems; the matter is important since it directly determines the definition of the TD temperature via (4.5). Most often (4D) $S(U) = k_B \log[\omega(U)]$ is used where $\omega(U)$ is the density of states but this has drawbacks (cf. [58]). The shortcomings of Boltzmann's entropy have been noticed since long ago and a slightly different definition of entropy is (4E) $S(U) = k_B \log[\Omega(U)]$ where $\Omega(U)$ is the integral (4F) $\Omega(U) = (1/h^{3N}) \int \prod d\mathbf{p}_i \int \prod d\mathbf{q}_i \Theta(U - H(\mathbf{q}, \mathbf{p}))$. An immediate advantage of (4E) is that the resulting expression for the temperature T defined by (4G) $(1/T) = dS/dU$ coincides with the notion of temperature used by the experimentalists; indeed (4H) $k_B T = \Omega(U)/\omega(U)$. For a harmonic oscillator the density of states $\omega(U)$ is a constant so (4H) implies $k_B T = U$ as desired (cf. [58] for more discussion).

5. TD AND FRACTALS

It seems presumptive to simply fractalize an arbitrary intensive or extensive TD variable in order to see what happens. But it does seem possible to “sneak” fractals into TD via entropy ideas and fractional calculus. For more on this theme we sketch from [22, 23] Thus the purpose in [22] is to point out that q-derivatives are naturally suited for describing systems with discrete dilatation symmetries such as fractal and multi-fractal sets, where the limit $q \rightarrow 1$ corresponds to continuous scale change. Thus the

q-derivative can be defined via

$$(5.1) \quad \partial_x^{(q)} f(x, y, \dots) = \frac{f(qx, y, \dots) - f(x, y, \dots)}{(q-1)x}$$

This derivative measures the rate of change with respect to a dilatation of its arguments by a factor of q . Even for some functions with no ordinary derivative a sequence of q 's going to 1 can be chosen in such a manner that the limit is well defined. We recall that self-similar sets are characterized by homogeneous functions, e.g. f is of degree ψ means

$$(5.2) \quad \partial_x^{(q)} f(x) = \frac{q^\psi - 1}{q - 1} \frac{f(x)}{x}$$

Following standard notation the q-number

$$(5.3) \quad [\psi]_q = \frac{q^\psi - 1}{q - 1} \Rightarrow x \partial_x^{(q)} f(x) = [\psi]_q f(x)$$

It is useful to note that the solution of **(5A)** $\partial_x^{(q)} f(x) = 0$ is either a constant or a function periodic in $\log(x)$, with period $\log(q)$, such that $A_q(qx) = A_q(x)$. With the modified product rule

$$(5.4) \quad \partial_x^{(q)} [g(x)f(x)] = g(qx)\partial_x^{(q)} f(x) + f(x)\partial_x^{(q)} g(x)$$

there is, besides the arbitrary additive constant of integration, an arbitrary multiplicative function of integration satisfying **(5B)** $f(x) = A_q(x)x^\psi$ (taking the constant of integration to be 0). The function $A_q(x)$ has some TD meaning, e.g. the free energy has the form **(5B)** and can be expressed in the form of a q-integral (cf. [21]). For $0 < q < 1$ the q-integral is defined as

$$(5.5) \quad \int_0^x D_t^{(q)} g(t) = (1-q)x \sum_0^\infty q^n g(q^n x); \quad \int_x^\infty D_t^{(q)} g(t) = (1-q) \sum_1^\infty q^{-n} g(q^{-n} x)$$

where $D_t^{(q)}$ denotes the q-differential. For any given $\psi > 0$ and $q \neq 1$ a function $A_q(x)$ with the desired property **(5A)** can be written via

$$(5.6) \quad \begin{aligned} A_q(x) &= x^{-\psi} \sum_{-\infty}^\infty \frac{g(q^n x)}{q^{n\psi}} = \\ &= \frac{1}{1-q} \int_0^x \frac{g(t)}{t^{1+\psi}} D_t^{(q)} + \frac{1}{1-q} \int_x^\infty \frac{g(t)}{t^{1+\psi}} D_t^{(q)} \end{aligned}$$

The function $g(x)$ is quite arbitrary (see [22] for more details). An example here is the Weierstrass-Mandelbrot function

$$(5.7) \quad f_b(x) = \sum_{-\infty}^\infty b^{n\psi} [1 - \cos(b^n x)]; \quad b > 1, \quad 0 < \psi < 1$$

The limit

$$(5.8) \quad \lim_{b \rightarrow 1} \frac{x \partial_x^{(b)} f_b(x)}{f_b(x)} = \psi$$

but $\lim_{b \rightarrow 1} x f'_b(x)/f_b(x)$ is not defined.

Now consider a fractal set S which is self-similar under scale changes by b and let $\rho_b(x)$ satisfy **(5A)** with $q = b$ (with respect to any one of its arguments x_k , $k = 1, \dots, d$ where d is the dimension of the embedding space. This function can be constructed as the $n \rightarrow \infty$ limit of a succession of $\rho_b^{(n)}$ such that

$$(5.9) \quad \rho_b^{(n)}(x) = \begin{cases} 1 & x \in \cup_i \Omega_i(b^n) \ (i = 1, \dots, N_n) \\ 0 & \text{otherwise} \end{cases}$$

where $\Omega_i(b^n)$ is a covering of the set S consisting of $N_n = N^n$ balls of linear size b^n ($b < 1$ and $N > 1$ is arbitrary). The fractal self-similarity dimension of this set is $d_f = \log(N)/\log(b^{-1})$ and to see this define

$$(5.10) \quad M(\ell_n, R) = \int_{\Gamma(R)} \rho_b^{(n)}(x) d^d x = \Phi_b(\ell_n, R) \ell_n^d (\ell_n/R)^{-d_f}$$

where $\ell_n = b^n \ell_0$. $\Gamma(R)$ is a region of linear size R of the embedding space of dimension d . The oscillatory amplitude $\Phi_b(\ell, R)$ is called lacunarity and is periodic in the logarithm of both ℓ and R , with period $\log(b)$. As an example consider the Cantor set with $d = 1$, $\ell_0 = 1$, $b = 1/3$, and $N = 2$. From (5.10) it is clear that $M(\ell_n, 1/3)/M(\ell_n, 1) = 1/2 = (1/3)^{d_f}$ yielding $d_f = \log(2)/\log(3)$. Now taking the limit of $n \rightarrow \infty$ one can define **(5D)** $M(R) = \lim_{n \rightarrow \infty} M(\ell_n, R) = \int_{\Gamma(R)} \rho_b(x) d^d x$. The usefulness

of this definition is that $M(R)$ is an eigenfunction of the operator $R \partial_R^{(q)}$ with $q = b$ being the scale change which takes the fractal set to itself such that **(5E)** $R \partial_R^{(b)} M(R) = [d_f]_b M(R)$, while its derivative fails to exist at an infinite number of points. It is convenient to introduce the dilatation operator such that $T_x^{(q)} f(x) = f(qx)$. Then $T_x^{(q)} = [1 + (q-1)x \partial_x^{(q)}]$ or $T_x^{(q)} = [\partial_x^{(q)}, x]$ where the brackets indicate the ordinary commutator. Then S is generated by an exact recursion relation based on a fine grain- ing of the covering by b , leading to **(5F)** $T_\ell^{(b)} M(\ell, R) = M(\ell_{n+1}, R)$ or **(5G)** $(T_\ell^{(b)})^k M(\ell, R) = M(b^k \ell, R)$. For small $b-1$ exponentiation yields

$$(5.11) \quad M(b^\ell \ell, R) = (T_\ell^{(b)})^k M(\ell, R) = e^{k[T_\ell^{(b)} - 1]} M(\ell, R)$$

There is also a discussion of multi-fractal situations with considerable detail.

We go now briefly to [56] for further connections of generalized entropies to Fisher information (with some repetition - cf. also [29]). First from [56]

note that the Renyi entropy isn't convex and does not have the property of finite entropy production, so no extension can have these properties either. There is some controversy about TD implications here; but this does not affect the meaning and applications in e.g. information theory. First write the Shannon entropy in the form **(5H)** $S_S(P) = -\sum_i p_i \log(p_i) = \sum p_i \log(1/p_i)$ while the Tsallis entropy is

$$(5.12) \quad S_T(P, q) = \frac{\sum_i p_i^q - 1}{1 - q} = \frac{1}{q - 1} \sum p_i \left[1 - p_i^{q-1} \right]$$

It is convenient here to use the notation

$$(5.13) \quad \log_q(x) = \frac{x^{1-q} - 1}{1 - q}; \quad e_q^x = [1 + (1 - q)x]^{1/(1-q)};$$

$$\log_q(xy) = \log_q x + \log_q y + (1 - q)(\log_q(x))(\log_q(y))$$

Exploiting this general notation the Tsallis entropy is **(5I)** $S_T(P, q) = -\sum p_i^q \log_q(p_i) = \sum p_i \log_q[1/p_i]$ which is sometimes called a q-deformed Shannon entropy. Then define an information measure via

$$(5.14) \quad I_i = I_i \left(\frac{1}{p_i} \right) = \log_q \left(\frac{1}{p_i} \right); \quad S_S(P) = \left\langle \log \left(\frac{1}{p_i} \right) \right\rangle_{lin};$$

$$S_T(P) = \left\langle \log_q \left(\frac{1}{p_i} \right) \right\rangle_{lin}$$

More generally, following Kolmogorov and Nagumo one can write **(5J)** $S = f^{-1}(\sum_i p_i f(I_i))$ with f a strictly monotone continuous function (KN function). Renyi instead showed that, if additivity is imposed on information measures, then the whole set of KN functions must reduce to two cases, namely **(5K)** $f(x) = x$ or $f(x) = c_1 b^{(1-q)x} + c_2$. Renyi's information-entropy measure is then

$$(5.15) \quad S_R(P, q) = \frac{1}{1 - q} \log_b \sum_i p_i^q = \frac{1}{1 - q} \log \sum_i p_i^q$$

(assume here $b = e$ is the logarithm base). In fact if one chooses in **(5K)** $c_1 = 1/(1 - q) - c_2$ then there follows

$$(5.16) \quad f(x) = \log_q e^x; \quad I_i = \log \left(\frac{1}{p_i} \right); \quad S_R(P, q) = \left\langle \log \left(\frac{1}{p_i} \right) \right\rangle_{exp}$$

(where $\langle \rangle$ is the KN average in (5.16)).

Now introduce the Sharma-Mittal and Supra-extensive entropies, beginning with the relation between Tsallis and Renyi entropies

$$(5.17) \quad S_R(P, q) = \frac{1}{1 - q} \log[1 + (1 - q)S_T(P, q)]$$

This is equivalent to

$$(5.18) \quad S_R(P, q) = \log \left[e_q^{S_T(P, q)} \right] \Rightarrow S_T(P, q) = \log_q \left[e^{S_R(P, q)} \right]$$

This suggests two possible further generalizations

$$(5.19) \quad S_{SM}(P, \{q, r\}) = \log_r e_q^{S_T(P, q)} = \frac{1}{1-r} \left[\left(\sum_i p_i^q \right)^{\frac{1-r}{1-q}} - 1 \right]$$

$$(5.20) \quad S_{SE}(P, \{q, r\}) = \log_q e_r^{S_R(P, q)} = \frac{\left[1 + \frac{1-r}{1-q} \log \sum_i p_i^q \right]^{\frac{1-r}{1-q}} - 1}{1-q}$$

(cf. [56] for more on this). Some rewriting is also possible via

$$(5.21) \quad \left(\sum_i p_i^q \right)^{1/(1-q)} \left(\sum_i p_i \left(\frac{1}{p_i} \right)^{1-q} \right)^{1/(1-q)} =$$

$$\left\langle \left(\frac{1}{p_i} \right)^{1-q} \right\rangle_{lin}^{1/(1-q)} = e_q^{<\log_q(1/p_i)>_{lin}} = e_q^{S_T(P, q)} = \left\langle \frac{1}{p_i} \right\rangle_{\log_q} = \Omega(P, q)$$

where the logarithmic mean $< \cdot >_{\log_q} \equiv \Omega(P, q)$ is used. One can then rewrite (5.21) in the form

$$(5.22) \quad S_T(P, q) = \log_q \Omega(P, q); \quad S_R(P, q) = \log \Omega(P, q);$$

$$S_{SM}(P, (q, r)) = \log_r \Omega(P, q); \quad S_{SE}(P, (q, r)) = \log_q e_r^{\Omega(P, q)}$$

$$(5.23) \quad \Omega(P, q) = \left\langle \frac{1}{p_i} \right\rangle_{\log_q} = \left(\sum p_i^q \right)^{1/(1-q)}$$

Recall also the Kullback-Leibler relative information-entropy measure ($SP \sim$ sample space)

$$(5.24) \quad S_{KL} = \int_{SP} p_1(x, \theta) \log \frac{p_2(x, \phi)}{p_1(x, \theta)} d^n x$$

and define then the information measures of Tsallis, Renyi, etc. as

$$(5.25) \quad S_T(\mathcal{P}, q) = \frac{1}{1-q} \left[\int p_1(x, \theta)^q p_2(x, \phi)^{1-q} d^n x - 1 \right]$$

$$S_R(\mathcal{P}, q) = \frac{1}{1-q} \log \int p_1(x, \theta)^q p_2(x, \phi)^{1-q} d^n x;$$

$$S_{SM}(\mathcal{P}(q, r)) = \frac{1}{1-r} \left[\left(\int p_1(x, \theta)^q p_2(x, \phi)^{1-q} d^n x \right)^{\frac{1-r}{1-q}} - 1 \right]$$

$$S_{SE}(\mathcal{P}, (q, r)) = \frac{\left[1 + \frac{1-r}{1-q} \log \int p_1(x, \theta)^q p_2(x, \phi)^{1-q}\right]^{\frac{1-r}{1-q}} - 1}{1 - q}$$

The Fisher information I_F is defined via

$$(5.26) \quad I_F(\mathcal{P}) = \int \frac{1}{p(x, \theta)} \left(\frac{\partial p(x, \theta)}{\partial \theta} \right)^2 dx$$

and for any other estimator one has a Cramer-Rao inequality **(5L)** $I_F e^2 \geq 1$. For n-dimensions the Fisher information matrix is defined via

$$(5.27) \quad F_{ij}(\theta) = \int \frac{1}{p(x, \theta)} \frac{\partial p(x, \theta)}{\partial \theta_i} \frac{\partial p(x, \theta)}{\partial \theta_j} d^n x$$

or as an expectation value **(5M)** $F_{ij}(\theta) = E(\partial_i \ell(\theta) \partial_j \ell(\theta))$ where $(\ell \sim$ partial derivatives). Note that symmetrization is needed by means of **(5N)** $\mathcal{D}(p_1, p_2) = \frac{1}{2} [S(p_1, p_1) + S(p_2, p_1)]$ to obtain

$$(5.28) \quad g_{ij}(\theta, \phi) = \frac{1}{2} \left[\frac{\partial^2 (S(\theta, \phi) + S(\phi, \theta))}{\partial \theta_i \partial \theta_j} \right]_{\phi=\theta}$$

Then write also **(5O)** $\mathcal{D}(\theta, \phi) = (1/2)[S(\theta, \phi) + S(\phi, \theta)]$ and it results that **(5P)** $g_{ij}^{SE}(\theta) = g_{ij}^{SM}(\theta) = g_{ij}^R(\theta) = g_{ij}^T(\theta) = qg_{KL}(\theta) = -qF_{ij}(\theta)$.

Next one establishes some relations between the various entropies mentioned above and the Fisher information. Thus set **(5Q)** $\partial_x \log_q(x) = 1/x^q$ and $\partial_x e_q^x = (e_q^x)^q$. Then

$$(5.29) \quad S_T(\theta, \phi) = \int p_1(x, \theta) \log_q \left(\frac{p_2(x, \phi)}{p_1(x, \theta)} \right) d^n x;$$

$$S_T(\phi, \theta) = \int p_2(x, \phi) \log_q \left(\frac{p_1(x, \theta)}{p_2(x, \phi)} \right) d^n x$$

The resulting (ij) derivatives are quite different than one would expect from (5.25) with identification only for normalized PDF. For distributions as in **(5N)** there is some simplification however and

$$(5.30) \quad [\partial_{ij} S_T(\theta, \phi)]_{\phi=\theta} = [\partial_{ij} S_T(\phi, \theta)]_{\phi=\theta} =$$

$$= -q \int \frac{1}{p(x, \theta)} \partial_i p(x, \theta) \partial_j p(x, \theta) d^n x = -qF_{ij}(\theta)$$

which via (14.28) yields $g_{ij}^T = -qF_{ij}(\theta)$ Similarly one has

- (1) **(5R)** $[\partial_{ij} S_R]_{\phi=\theta} = [\partial_{ij} S_T]_{\phi=\theta} \Rightarrow g_{ij}^R = g_{ij}^T = -qF_{ij}$
- (2) **(5S)** $[\partial_{ij} S_{SM}]_{\phi=\theta} = [\partial_{ij} S_T]_{\phi=\theta} = -qF_{ij}$
- (3) **(5T)** $g_{ij}^{SE} = g_{ij} S_T = -qF_{ij}$

Thus the Fisher information accounts (modulo $-q$) for the change of multiplicity (or the number of microstates) under a statistical parameter variation.

6. Q-FISHER AND TSALLIS

We follow here [29] (JMP 50 (2009), -13303) (cf. also [2, 5, 18, 21, 22, 29, 30, 31, 33, 38, 39, 42, 49, 52, 53, 56, 64, 65, 68, 73, 76, 80]). Denote the q-logarithm and exponential via

$$(6.1) \quad \log_q(x) = \frac{x^{1-q} - 1}{1-q}; \quad \exp_q(x) = \begin{cases} [1 + (1-q)x]^{1/(1-q)} & 1 + (1-q)x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and note that

$$(6.2) \quad \exp_q[x + y + (1-q)xy] = \exp_q(x)\exp_q(y);$$

$$\log_q(xy) = \log_q(x) + \log_q(y) + (1-2)\log_q(x)\log_q * y$$

The set of all probability density functions on (\mathbf{R}) is denoted by **(6A)** $D = \left\{ f : \mathbf{R} \rightarrow \mathbf{R}; f(x) \geq 0; \int_{-\infty}^{\infty} f(x)dx = 1 \right\}$ and the Tsallis relative entropy is $H_q(\phi(x)) = -\int_{-\infty}^{\infty} \phi(x)^q \log_q \phi(s) dx$. Thus

$$(6.3) \quad D_q(\phi(x)|\psi(x)) = \int_{-\infty}^{\infty} \phi(x)^q [\log_q(\phi(x)) - \log_q(\psi(x))] dx$$

For $q \rightarrow 1$ the Tsallis q-entropy converges to the Shannon entropy and the relative entropy converges to the Kullback-Leibler (KL) divergence $(\bullet) D_1(\phi(x)|\psi(x)) = \int_{-\infty}^{\infty} \phi(x)[\log(\phi(x)) - \log(\psi(x))] dx$. Define now two constraints on the normalized q-expectation value and q-variance, namely

$$(6.4) \quad C_q^{(c)} = \left\{ f \in D; \frac{1}{c_q} \int_{-\infty}^{\infty} x f(x)^q dx = \mu_q \right\};$$

$$C_q^{(g)} = \left\{ f \in C_q^{(c)}; \frac{1}{c_q} \int_{-\infty}^{\infty} (x - \mu_q)^2 f(x)^q dx = \sigma_q^2 \right\}$$

The q-canonical distribution $\phi_q^{(c)}(x) \in D$ and the q-Gaussian distribution $\phi_q^{(g)} \in D$ are

$$(6.5) \quad \phi_q^{(c)}(x) = \frac{1}{Z_q^{(c)}} \exp_q \left[-\beta_q^{(c)}(x - \mu_q) \right]; \quad Z_q^{(c)} = \int_{-\infty}^{\infty} \exp_q \left[-\beta_q^{(c)}(x - \mu_q) \right];$$

$$\phi_q^{(g)}(x) = \frac{1}{Z_q^{(g)}} \exp_q \left[\frac{-\beta_q^{(g)}(x - \mu_q)^2}{\sigma_q^2} \right]; \quad Z_q^{(g)} = \int_{-\infty}^{\infty} \exp_q \left[\frac{-\beta_q^{(g)}(x - \mu_q)^2}{\sigma_q^2} \right]$$

One shows next that **(6B)** $D_q(\phi(x)|\psi(x)) \geq 0$ with equality if and only if $\phi(x) = \psi(x)$ for all x . Further

$$(6.6) \quad H_q(\phi(x)) \leq -c_q \frac{1}{Z_q^{(c)}}; \quad \text{equality} \iff \phi(x) = \frac{1}{Z_q^{(c)}} \exp_q \left[-\beta_q^{(c)}(x - \mu_q) \right]$$

(cf. [29] for proof). The generalized free energy takes a minimum

$$(6.7) \quad F_q = \mu_q - \frac{1}{\beta_q^{(c)}} H_q(\phi(x)) \geq \mu_q + \frac{c_q}{\beta_q^{(c)}} \log_q \frac{1}{Z_q^{(c)}}$$

if and only if $\phi(x) = (1/Z_q^{(c)}) \exp_q[-\beta_q^{(c)}(x - \mu_q)]$. Further if $\phi \in C_1^{(c)}$ then $H_1(\phi(x)) \leq \log(Z_1^{(c)})$ with equality $\iff \phi(x) = (1/Z_1^{(c)}) \exp[-\beta_1^{(c)}(x - \mu)]$. Next it can be shown that for $0 \leq q < 3$ and $q \neq 1$, if $\phi \in C_q^{(g)}$

$$(6.8) \quad H_q(\phi(x)) \leq -c_q \log_q \frac{1}{Z_q^{(g)}} + c_q \beta_q^{(g)} [Z_q^{(g)}]^{q-1}$$

with equality if and only if

$$(6.9) \quad \phi(x) = \frac{1}{Z_q^{(g)}} \exp_q \left[-\frac{\beta_q^{(g)}(x - \mu_q)^2}{\sigma_q^2} \right]$$

where $Z_q^{(g)} = \int_{-\infty}^{\infty} \exp_q \left[-\beta_q^{(g)}(x - \mu_q)^2 / \sigma_q^2 \right] dx$ with $\beta_q^{(g)} = 1/(3 - q)$.

Now the fact that the Gaussian distribution minimizes the Fisher information leads to the study the Tsallis distribution (q-Gaussian) to see if it minimizes a q-Fisher information as a 1-parameter extension. In this direction one defines the q-score function $s_q(x)$ and the q-Fisher information $J_q(x)$ via

$$(6.10) \quad s_q(x) = \frac{d \log_q f(x)}{dx}; \quad J_q(X) = E_q[s_q(x)^2]$$

where E_q is defined as $E_q[g(X)] = \frac{\int g(x) f(x)^q dx}{\int f(x)^q dx}$ for random variables $g(X)$, with continuous $g(x)$ and a probability density $f(x)$ (note the definition of J_q differs from [68] for example). Then as an example for a random variable G obeying a q-Gaussian distribution consider

$$(6.11) \quad \phi_q^{(g)}(x) = \frac{1}{Z_q} \exp_q \left[-\frac{\beta_q(x - \mu_q)^2}{\sigma_q^2} \right]$$

where $\beta_q = 1/(3 - q)$ and the q-partition function is $Z_q = \int_{-\infty}^{\infty} \exp_q[(x - \mu_q)^2 / \sigma_q^2] dx$, with q-score function and q-Fisher information

$$(6.12) \quad s_q(x) = -\frac{2\beta_q Z_q^{q-1}}{\sigma_q^2} (x - \mu_q); \quad J_q(G) = \frac{4\beta_q^2 Z_q^{2q-2}}{\sigma_q^2}$$

so $\lim_{q \rightarrow 1} J_q(G) = 1/\sigma_1^2$. Then given the random variable X with probability density function $p(x)$, the q-expectation value $\mu_q = E_q(X)$, and the

q-variance $\sigma_q^2 = E_q(X - \mu_q)^2$. There is then a q-Cramer-Rao inequality (6.13)

$$J_q(X) \geq \frac{1}{\sigma_q^2} \left[\frac{2}{\int p(x)^q dx} - 1 \right]; \quad q \in [0, 1] \cup (1, 3) \Rightarrow J_q(X) \geq \frac{1}{\sigma_q^2} \quad (q \in (1, 3))$$

To see this assume $\lim_{x \rightarrow \pm\infty} f(x)p(x) = 0$ for any $q \geq 0$, any probability density p and any smooth function f suitably behaved at $\pm\infty$. Then using p in the score function (6.10)

$$(6.14) \quad E_q(X - \mu_q)s_q(x) = \frac{\int (x - \mu_q)p(x)^2 s_q(x) dx}{\int p(x)^q dx} = \frac{\int (x - \mu_q)p'(x) dx}{\int p(x)^q dx} = \frac{-1}{\int p(x)^q dx}$$

Note for $q < 1$ there is no relation between $J_q(X)$ and $1/\sigma_q^2$ beyond (6.13). Thus it seems that the Tsallis entropies and q-Fisher information make sense only for the case of $q \geq 1$ in the present setting.

REMARK 6.1. We mention here for example the papers [3, 17, 19, 25, 39, 47, 50, 54, 55, 64, 67, 66, 68, 78, 79], involving q-entropy, q-information, q-TD, Fisher information, the Frieden-Soffer idea of extreme physical information, the SE, non-equilibrium TD, Legendre structure, etc. They seem to signal new portals between TD, QM, and gravitational physics to go along with the cosmological input of Padmanabhan [61, 62, 63] and others connecting gravity and entropy (see e.g. [40, 77] and cf. also [8, 9, 10, 11] for fractal structure). We mention also a series of papers [34, 70, 71, 72] connecting macro (TD) physics and micro (QM) physics in a very interesting manner and would like to refer as well to some earlier work involving TD and QM in e.g. [1, 13, 15, 16, 35, 36]. ■

REFERENCES

- [1] D. Acosta, P. Fernandez de Cordoba, J. Isidro, and J. Santander, hep-th 1107.1898; Jour. Physics, Conf. Series 361 (2012), 01202; math-ph 1206.4941
- [2] E. Akkermans, C. Dunne, and A. Teplyaev, cond-mat 1010.1148
- [3] J. Batle, M. Casas, A. Plastino, and A.R. Plastino, quant-ph 0302143
- [4] S. Bayin, math-ph 1203.4556
- [5] J. Bercher, math-ph 1203.1435 and 1206.0567
- [6] D. Bonatsos and C. Daskaloyannis, Prog. Part. Nuclear Physics, 43 (1999), 537-618
- [7] P. Bracken, gr-qc 1111.5068
- [8] G. Calcagni, hep-th 0912.3142, 1001.0571, 1012.1244, 1106.0295, 1204.2550, and 1205.5046
- [9] G. Calcagni and G. Nardelli, math-ph 1202.5383; hep-th 1004.5144
- [10] G. Calcagni, G. Nardelli, and M. Scalisi, hep-th 1207.4473
- [11] G. Calcagni, S. Gielen, and D. Oriti, gr-qc 1201.4151
- [12] R. Carroll, Calculus revisited, Kluwer, 2002
- [13] R. Carroll, Fluctuations, information, gravity, and the quantum potential, Springer, 2006; On the quantum potential, Arima Publ., 2007; On the emergence theme of physics, World Scientific, 2010
- [14] R. Carroll, Progress in Physics, 1 (2012), 27-29 (gr-qc 1110.3059) and 2 (2012), 82-86 (math-ph 1206.0900)
- [15] R. Carroll, Quantum Potential as Information: A mathematical survey, in: New trends in quantum information, Aracne Edit., 2010, pp. 155-189
- [16] R. Carroll, gr-qc 1104.0383 (J. Physics, Conf. Series, 361 (2012), 012010)
- [17] L. Chimento, F. Pennini, and A. Plastino, physics 1208.3671; math-ph 1207.3377; cond-mat 0005006
- [18] W. da Cruz, cond-mat 9811007
- [19] E. Curado and A. Plastino, cond-mat 0601076
- [20] R. Earnshaw and E. Riley (editors), Brownian motion ..., Nova Publ., 2011, pp. 1-69
- [21] A. Ezan, Phys. Lett. A, 225 (1997), 235-238
- [22] A. Erzan and J. Eckmann, Phys. Rev. Lett., 78 (1997), 3245-3248
- [23] A. Erzan and A. Gorbunov, cond-mat 98078052
- [24] K. Falconer, The geometry of fractal sets, Cambridge Univ. Press, 1985
- [25] S. Flego, A. Plastino, and A.R. Plastino, math.ST 1101.4659; cond-mat 1101.4661; quant-ph 1105.5054
- [26] B. Frieden, Physics from Fisher information, Cambridge Univ. Press, 1998; Science from Fisher information, Springer, 2004
- [27] B. Frieden and R. Gatenby, Exploratory data analysis using Fisher information, Springer, 2007
- [28] B. Frieden and B. Soffer, quant-ph 0609157
- [29] S. Furuichi, cond-mat 1001.1383, 1010.0392, and 1104.0360; Jour. Phys. Conf. Ser., 201 (2010), 012016
- [30] S. Furuichi and F. Mitroi, cond-mat 1104.5603
- [31] S. Furuichi, and K. Yanagi, quant-ph 1010.0392
- [32] M. Gell-Mann and C. Tsallis, Nonextensive entropy, Oxford Univ. Press, 2004
- [33] C. Godinho, J. Weberszpil, and J. Helayei-Neto, CSF, 45 (2012) (765-771)
- [34] O. Golubeva, A. Sukhanov, and V. Baryakhtar, cond-mat 1112.1598
- [35] G. Grössing, J. Pascasio, and H. Schwabl, Found. Phys., 41 (2011), 1437

- [36] G. Grössing, S. Fussy, J. Pascasio, and H. Schwabl, quant-ph 1106.5994, 1005.1058, and 1004.4596; Jour. Phys., Conf. Series, 361 (2012), 012008; quant-ph 1205.3393
- [37] R. Herrmann, math-ph 0510099; physics 0711.3701, 0805.3434, and 1007.1084; math.GM 1107.4205; Fractional calculus, World Scientific, 2011; Fraktionale Infinitesimalrechnung, Altenstadt, 2008
- [38] A. Hernando, D. Puigdomenech, D. Villaendas, and C. Versperinas, cond-mat 0902.2738
- [39] A. Hernando, C. Versperinas, and A. Plastino, physics 0908.0504
- [40] T. Jacobson, Phys. Rev. Lett., 75 (1995), 1260-1263
- [41] M. Jeng, S. Xu, E. Hawkins, and J. Schwarz, math-ph 0810.1543
- [42] P. Jizba and T. Arimitsu, cond-mat 0108185, 0207707, 0510092, and 0312034
- [43] G. Jumarie, Applied Math. Letters, 18 (2005), 739-748 and 817-826, 22 (2009), 378-385 and 1659-1664; Chaos, Solitons, and Fractals, 32 (2007), 969-987; 40 (2009), 1428-1448 and 41 (2009), 1590-1604; 12 (2001), 2577-2587; Acta Math. Sinica, 28 (2012), 1741-1768
- [44] V. Kac and P. Cheung, Quantum calculus, Springer, 2002
- [45] A. Kilbas, H. Srivastava, and J. Trujillo, Theory and applications of fractional differential equations, North-Holland (Elsevier), 2006
- [46] J. Klafter, S. Lim, and R. Metzler, Fractional Dynamics, World Scientific, 2012
- [47] A. Klimyk, math-ph 0508032
- [48] V. Koblelev, math-ph 1292.274; Chaos, 16 (2006), 043117
- [49] L. Kobelov, hep-th 0002005, physics 0002008, 0003036, 0005068, 0006043, 0011036, 0011038, 0104039, and 0104040
- [50] A. Kowalski, M. Martin, A. Plastino, and L. Zunino, cond-mat 08124221
- [51] N. Laskin, math-ph 1009.5533 and 0811.1769; Phys. Rev. E, 66 (2002), 056108
- [52] R. Lopez-Ruiz, nlin 0312056
- [53] B. Lopez-Ruiz, A. Nagy, E. Romera, and J. Sanudo, quant-ph 0905.3360
- [54] S. Martinez, F. Pennini, A. Plastino, and H. Vucetich, cond-mat 0105355
- [55] S. Martinez and A. Plastino, cond-mat 0310263
- [56] M. Masi, cond-mat 0611300
- [57] M. Naber, math-ph 0410028
- [58] J. Naudts, cond-mat 0911.5392; Entropy, 10 (2008), 131-149
- [59] L. Nottale, physics, 0812.0941, 0812.3857, and 0901.1270; Chaos, solitons, and fractals, 25 (2005), 797-803; 7 (1996), 877-938, 10 (1999), 459
- [60] P. Nottale, Fractal space-time and microphysics: Towards a theory of scale relativity, World-Scientific, 1993; Scale relativity and fractal space-time, Imperial College Press, 2011
- [61] T. Padmanabhan, gr-qc 0202078, 0202080, 0205090, 0302068, 0308070, and 0311036; hep-th 0204019, 0205278, 0408051,
- [62] T. Padmanabhan, hep-th 0910.0839, 0911.1403, 0911.5004, 0912.3165, 1012.4476, and 1205.5683; gr-qc 1001.3380 and 1206.4916; astro-ph 1207.0505
- [63] T. Padmanabhan, gr-qc 1003.5665, 1012.0119 and 1109.3846
- [64] F. Pennini and A. Plastino, cond-mat 0311586, 0312680, 0405033, and 0407071
- [65] F. Pennini, A. Plastino, and G. Ferri, cond-mat 0408072
- [66] A. Plastino, A.R. Plastino, and B. Soffer, cond-mat 0509697
- [67] A.R. Plastino and C. Tsallis, cond-mat 1205-6048
- [68] M. Portesi, A. Plastino, and F. Pennini, cond-mat 0510434
- [69] E. Romera, R. Lopez-Ruiz, J. Sanudo, and A. Nagy, nlin.PSI 0901.1752
- [70] A. Sukhanov, cond-mat 0509113; Teor. Math. Phys., 154 (2008), 153-164

- [71] A. Sukhanov and O. Golubeva, Teor. Math. Phys., 160 (2009), 1177-1189; quant-ph 1012.5056
- [72] A. Sukhanov, V. Baryakhtar, and O. Golubeva, cond-mat 1012.5054 and 1112.1876
- [73] H. Suyuri, cond-mat 0502298
- [74] V. Tarasov, physics 0602096, 0604491, 0804.0586, 0907.2699, 1107.4205, 1107.5682, and 1107.5749
- [75] C. Tsallis, Jour. Stat. Phys., 52 (1988), 479-487
- [76] M. Ubriaco, hep-th 9305046; cond-mat 9904079, 0902.2726, 0907.1970, and 1205.2591
- [77] E. Verlinde, hep-th 1001.0785
- [78] C. Vignat and A. Plastino, cond-mat 0505158
- [79] C. Vignat, A. Plastino, and A.R. Plastino, cond-mat 1011.6367
- [80] J. Weberszpil, C. Godinho, A. Cherman, and J. Helayei-Neo, math-ph 1206.2513